A GEOMETRIC INVESTIGATION INTO METABELIAN GROUPS GENERATED BY FOUR ELEMENTS OF ORDER P

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MARGARET MARY HANSMAN

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Let G be a metabelian group of order p^m which is generated by n elements, $u_1, u_2, \cdots\cdots u_n$, of order p (p a prime) and all of whose elements but I are of order p. Let the commutator subgroup K of G coincide with the central which is of order p^c, and let G contain a maximal abelian subgroup $\{K,u\}$ of order p^{c+1}. At most

$$c = \frac{n(n-1)}{2}$$

and

$$m = \frac{n(n+1)}{2} \quad .$$

If there are relations among the s_{ij} where s_{ij} is the commutator of u_i and u_j, c and m are smaller, and the group is a quotient group of G by a subgroup D of K.

For n greater than 3, K contains both commutators and non-commutators. The general element of K,

$$s_{12}^{\alpha_{12}} s_{13}^{\alpha_{13}} \cdots\cdots s_{n-1\ n}^{\alpha_{n-1\ n}} \quad ,$$

is a commutator if and only if

$$(1) \qquad \alpha_{ij}\alpha_{kl} - \alpha_{ik}\alpha_{jl} + \alpha_{il}\alpha_{jk} \equiv 0 \pmod{p},$$

$$i,j,k,l = 1,2,\cdots\cdots,n$$
$$i < j < k < l \quad .$$

A correspondence may be set up between elements of G and points of S_{n-1}. To the general element of G,

$$u_1^{k x_1} u_2^{k x_2} \cdots\cdots u_n^{k x_n} c,$$

where u_i are the generators of G and c belongs to K and k is any rational number, corresponds the unique point $(x_1, x_2, \cdots\cdots, x_n)$. Similarly a correspondence may be set up between elements of K and points of a space of $\frac{(n-2)(n+1)}{2}$ dimensions in which the point $(\alpha_{12}, \alpha_{13}, \cdots\cdots \alpha_{n-1\ n})$ corresponds to the elements

$$s_{12}^{k\alpha_{12}} s_{13}^{k\alpha_{13}} \cdots\cdots s_{n-1\ n}^{k\alpha_{n-1\ n}}$$

for all rational k; the point lies on the manifold (1) if and only if the element is a commutator.

Any non-singular collineation of S_{n-1} induces on this commutator space a non-singular collineation under which (1) is invariant. Hence under the corresponding isomorphisms of G with itself it is found that commutators go into commutators and non-commutators into non-commutators.

If n is 4, elements of G correspond to points of ordinary three-space, and elements of K to points of five-space. Also in this case, (1) reduces to the single congruence

$$(2) \qquad \alpha_{12}\alpha_{34} - \alpha_{13}\alpha_{24} + \alpha_{14}\alpha_{23} \equiv 0 \pmod{p}.$$

This is the condition that (α) be Plücker coordinates of a line in S_3, and it may be shown that this line joins pairs of points which correspond to elements of G having for their commutator the element of K corresponding to (α). Hence every commutator may be represented either by a line of S_3 or by a point of S_5 on (2).

If D is cyclic, it is generated by one element of K and may be represented by a point of five-space. There are two resulting groups G depending on whether the point lies on (2) or not. In general, if D is of order p^r, it is generated by r elements of K and may be represented in S_5 by a linear space of r-1 dimensions. There are various ways in which the subspace may intersect (2) and each of these ways gives rise to a group G. There follows a list of possible groups G. Where there are two groups of the same order, they are distinguished by the number of abelian subgroups of order p^{c+2} which they contain. There is also given for each a set of relations which together with

$$\begin{cases} G = \{u_1, u_2, u_3, u_4\} \\ u_j^{-1} u_i u_j = u_i s_{ij} \qquad i,j = 1,2,3,4 \qquad i < j \end{cases}$$

make up a canonical form for the generation of the group. The group represented by any linear (r-1)-space in S_5 is isomorphic to that one of correct type listed.

Order of G	Order of D	S_{r-1}	Relation to (2)	Relations on commutators	Abelian subgroups of order p^{c+2}
p^9	p	Point	On (2)	$s_{34}=I$	$\{K, u_3, u_4\}$
p^9	p	Point	Not on (2)	$s_{12}=s_{34}$	None
p^8	p^2	Line	on (2)	$s_{24}=s_{34}=I$	$\{K, u_4, u_2^m u_3^n\}$ m,n not both 0 ($p+1$ of them)
p^8	p^2	Line	Tangent to (2)	$s_{34}=I$ $s_{13}=s_{24}$	$\{K, u_3, u_4\}$
p^8	p^2	Line	Bisecant of (2)	$s_{12}=s_{34}=I$	$\{K, u_1, u_2\}$ $\{K, u_3, u_4\}$
p^8	p^2	Line	Does not touch (2)	$s_{12}=s_{34}$ $s_{24}=s_{13}^r$ (r not a square)	None
p^7	p^3	Plane	on (2)	$s_{23}=s_{24}=s_{34}=I$	$\{K,H\}$ (p^2+p+1 of them. H is a subgroup of order p^2 of $\{u_2,u_3,u_4\}$
p^7	p^3	Plane	Tangent to (2) along a line	$s_{24}=s_{34}=I$ $s_{14}=s_{23}$	$\{K, u_4, u_2^m u_3^n\}$ m,n not both 0 ($p+1$ of them)
p^7	p^3	Plane	Intersects (2) in a degenerate conic	$s_{13}=s_{24}=s_{34}=I$	$\{K, u_4, u_2^m u_3^n\}$ $\{K, u_3, u_1^c u_4^d\}$ ($2p+1$ of them)
p^7	p^3	Plane	Tangent at a point	$s_{12}=s_{34}$ $s_{13}^r=s_{24}$ $s_{23}=I$	$\{K, u_2, u_3\}$
p^7	p^3	Plane	Intersects (2) in a non-degenerate conic	$s_{12}=s_{34}=I$ $s_{13}=s_{24}$	$\{K, u, v\}$ where $u=u_1^{sbx_1} u_2^{sdx_2} u_3^{rdx_2} u_4^{rbx_1}$ $v=u_1^{sax_1} u_2^{scx_2} u_3^{rcx_2} u_4^{rax_1}$

At first glance it would seem that there is another group in the case of a plane on (2): i.e. that generated by

$$s_{12}=s_{13}=s_{14}=I$$

but this group contains no maximal subgroup of order p^{c+1}, and thus does not satisfy the required conditions.

To show that any group is isomorphic to one of these eleven canonical forms, it is sufficient to show that a collineation on S_3 may be found which transforms the subspace representing the subgroup D into that special one given in the table.

The reference system in S_5 is determined by six points on (1): $A_{12}(1,0,0,0,0,0)$, ... , $A_{34}(0,0,0,0,0,1)$ which correspond to the edges of the tetrahedron of reference in S_3: $A_1(1,0,0,0)$, ... , $A_4(0,0,0,1)$. It is easily seen that the line joining two reference points which have a subscript in common lies on (2), and that joining two with no subscript in common is a bisecant of (2). Any three having a subscript in common lie on a plane of (2) which corresponds to a bundle of lines in three-space, and any three whose subscripts are merely permutations of three numbers two at a time lie on a plane of (2) corresponding to the lines of a plane in three-space. These two types of planes will be called α-planes and π-planes respectively. Two planes of the same type intersect in a point and two planes of different types either do not intersect at all or have a line of points in common.

In the first case the given point should be chosen for A_{34}. There is great freedom in choosing the rest of the reference system. A_{12} may be taken as any point of (2) which does not lie in the polar prime of A_{34}, since their join is to be a bisecant of (2). A_{24} is any point of (2) in the intersection of the polar primes of both these points, while A_{13} also must lie in this intersection but not in the polar prime of A_{24}. A_{23} must lie in the α-plane through $\overline{A_{12}A_{24}}$ and the π-plane through $\overline{A_{12}A_{13}}$; since these planes intersect there must be a line of points possible. The final reference point A_{14} is then fixed as the intersection of the α-planes through $\overline{A_{12}A_{13}}$ and $\overline{A_{24}A_{34}}$.

Now suppose the point does not lie on (2). It may be shown that it lies on bisecants of (2). The intersections of the bisecant which is chosen may be taken for A_{12} and A_{34}. The other four points are found as above.

If D is represented by a line of (2), two points of the line should be made A_{24} and A_{34}.

For a tangent line let the point of tangency be A_{34}. Consider another point of the line; through it must be drawn a bisecant whose intersections with (2) will be chosen for A_{13} and A_{24}. Hence there must be found a point of (2) which lies in the polar prime of A_{34} but not in that of the second point of the line. This will be taken for one of the two required points and determines the bisecant.

If the line is a bisecant the procedure is exactly as in the case of a point not on (2).

Suppose the line does not touch (2). It is necessary to draw bisecants through two points of the line whose intersections may be made A_{12} and A_{34}, and A_{13} and A_{24} respectively. Now these points are such that the two bisecants are conjugate to each other with respect to the surface. Hence the two points of the line must be chosen conjugate to each other. Any point of the line may be taken for one, but the other is then fixed. When they have been chosen, find a point of (2) which is in the polar prime of the second but not in that of the first. This point and the first point of the conjugate pair determine the first bisecant, say $A_{12}A_{34}$. By choice of proportionality factor, this makes the first point take on the form (1,0,0,0,0,-1). For A_{24} find a point of (2) which lies in the polar primes of the two reference points already found but not in that of the second point of the conjugate pair. It fixes the second bisecant. The second point now has the form (0,r,0,0,-1,0). Since this line does not touch (2) it may be seen by direct substitution in the congruence defining (2) that r must not be congruent to a square modulo p. As a matter of fact it may be made any not-square in the field.

Next consider planes on (2). As was seen above, only π -planes give groups of the required type. Choose any three non-collinear points of the plane for A_{23}, A_{24}, A_{34}.

Suppose the plane is tangent along a line. Two points of the line may be used for A_{24} and A_{34}. Consider any point of the plane not on the line. A bisecant through it must be found for $\overline{A_{14}A_{23}}$. A_{23} must lie in the π-plane through the line of tangency, and not in the polar prime of the third point of the plane. Since the points common to these two spaces consist only of those on the line of tangency, any point of the π-plane may be chosen for A_{23}.

If the intersection is a degenerate conic take the double point of the conic for A_{34}, and any other two points on the two lines respectively for A_{13} and A_{24}.

Let the plane be tangent at a point. The point of tangency may be taken for A_{23}. Next choose a line of the plane which lies in the polar prime of A_{23}. Take a conjugate pair of points of the line and proceed as in the case of the line not touching the surface.

If the plane intersects in a non-degenerate conic, choose any pair of points of the conic for A_{12} and A_{34}. The pole of $\overline{A_{12}A_{34}}$ with respect to the conic may be made (0,1,0,0,-1,0), Choose a bisecant through the point in the polar of $\overline{A_{12}A_{14}}$ with respect to (2). Its intersections may be taken as A_{13} and A_{24}.

No higher dimensioned subspaces give groups with maximal abelian subgroups of order p^{c+1}. A three-space intersects every plane of (2) in a point at least. Hence in every bundle of lines in S_3 there is a line whose corresponding point on (2) lies on the three-space. Thus every element of G is permutable with some other element of G and there is no maximal abelian subgroup of order p^{c+1}.

By extending the theory of line coordinates to spaces of dimension higher than four, this method may be applied to groups G with more than four generators.

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Margaret Mary Hansman was born October 16, 1911 in Colorado Springs, Colorado. She attended the public schools there and in the fall of 1927 entered Colorado College in that city. In June 1931 she received the A.B. degree. Three years later she returned to Colorado College to study for the master's degree, which she received in 1936. The next year she was given a Teaching Assistantship at the University of Illinois, and spent the next four years there working for the degree of Doctor of Philosophy.